SN and redex creation in higher-order typed λ-calculus

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Plan

- Higher-order typed λ-calculus
- Weak vs Strong normalization
- Redex creation and strong normalization
- Girard's proof for strong normalization
- Finite developments
- Open problem

1st&2nd-order typing rules

(variable)
$$\frac{}{\Gamma, x{:}\tau \vdash x{:}\tau}$$

(application)
$$\frac{\Gamma \vdash M : \sigma \to \tau \qquad \Gamma \vdash N{:}\sigma}{\Gamma \vdash MN : \tau}$$

(abstraction)
$$\frac{\Gamma, x{:}\sigma \vdash M{:}\tau}{\Gamma \vdash \lambda x.M : \sigma \to \tau}$$

(1st-order typing)

$$\frac{\Gamma \vdash M : \forall\alpha.\tau}{\Gamma \vdash M : \tau\{\alpha := \sigma\}}$$

$$\frac{\Gamma \vdash M{:}\tau \qquad \alpha \notin \mathrm{TVar}(\Gamma)}{\Gamma \vdash M : \forall\alpha.\tau}$$

higher-order typing rules

(axioms) $\quad <>\vdash c : s,\quad$ if $(c : s) \in \mathcal{A}$;

(start) $\quad \dfrac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A},\quad$ if $x \equiv {}^s x \notin \Gamma$;

(weakening) $\quad \dfrac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B},\quad$ if $x \equiv {}^s x \notin \Gamma$;

(product) $\quad \dfrac{\Gamma \vdash A : s_1 \quad \Gamma, x{:}A \vdash B : s_2}{\Gamma \vdash (\Pi x{:}A.B) : s_3},\quad$ if $(s_1, s_2, s_3) \in \mathcal{R}$;

(application) $\quad \dfrac{\Gamma \vdash F : (\Pi x{:}A.B) \quad \Gamma \vdash a : A}{\Gamma \vdash Fa : B[x := a]};$

(abstraction) $\quad \dfrac{\Gamma, x{:}A \vdash b : B \quad \Gamma \vdash (\Pi x{:}A.B) : s}{\Gamma \vdash (\lambda x{:}A.b) : (\Pi x{:}A.B)};$

(conversion) $\quad \dfrac{\Gamma \vdash A : B \quad \Gamma \vdash B' : s \quad B =_\beta B'}{\Gamma \vdash A : B'}.$

Usual sorts

$$(s_1, s_2, s_3) = (s_1, s_2, s_2)$$

where (s_1, s_2) possible values are:

$\lambda\!\rightarrow$	$(*,*)$			
$\lambda 2$	$(*,*)$	$(\Box,*)$		
$\lambda\mathrm{P}$	$(*,*)$		$(*,\Box)$	
$\lambda\mathrm{P}2$	$(*,*)$	$(\Box,*)$	$(*,\Box)$	
$\lambda\underline{\omega}$	$(*,*)$			$(\Box,\Box)$
$\lambda\omega$	$(*,*)$	$(\Box,*)$		$(\Box,\Box)$
$\lambda\mathrm{P}\underline{\omega}$	$(*,*)$		$(*,\Box)$	$(\Box,\Box)$
$\lambda\mathrm{P}\omega{=}\lambda\mathrm{C}$	$(*,*)$	$(\Box,*)$	$(*,\Box)$	$(\Box,\Box)$

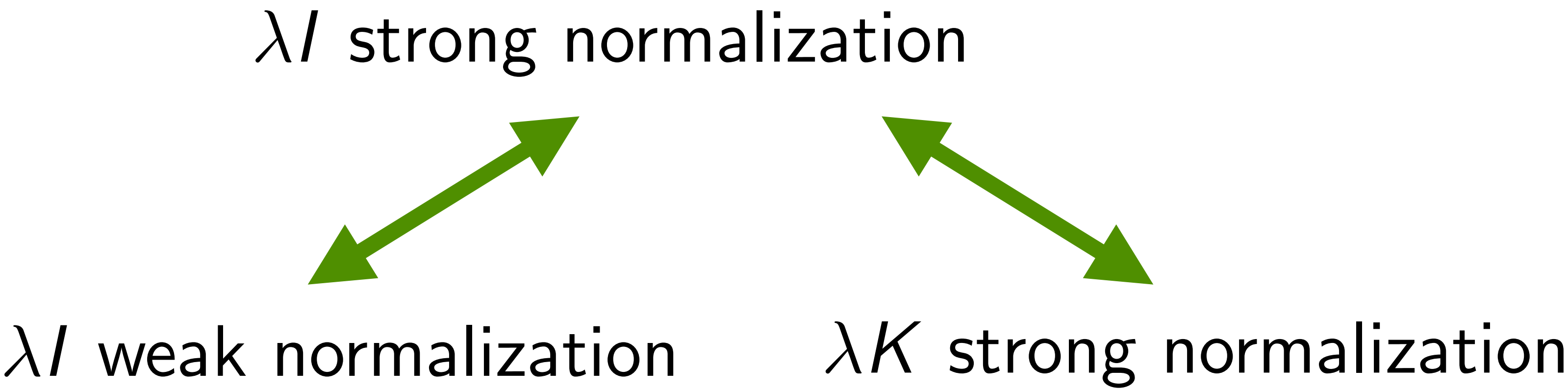
Usual abbrevs

$$\forall\alpha.A \equiv \Pi\alpha{:}{*}.A$$

$$\Lambda\alpha.M \equiv \lambda\alpha{:}{*}.M$$

$$A \to B \equiv \Pi\, x : A \,.\, B \quad \text{when } x \notin \mathsf{FVar}(B)$$

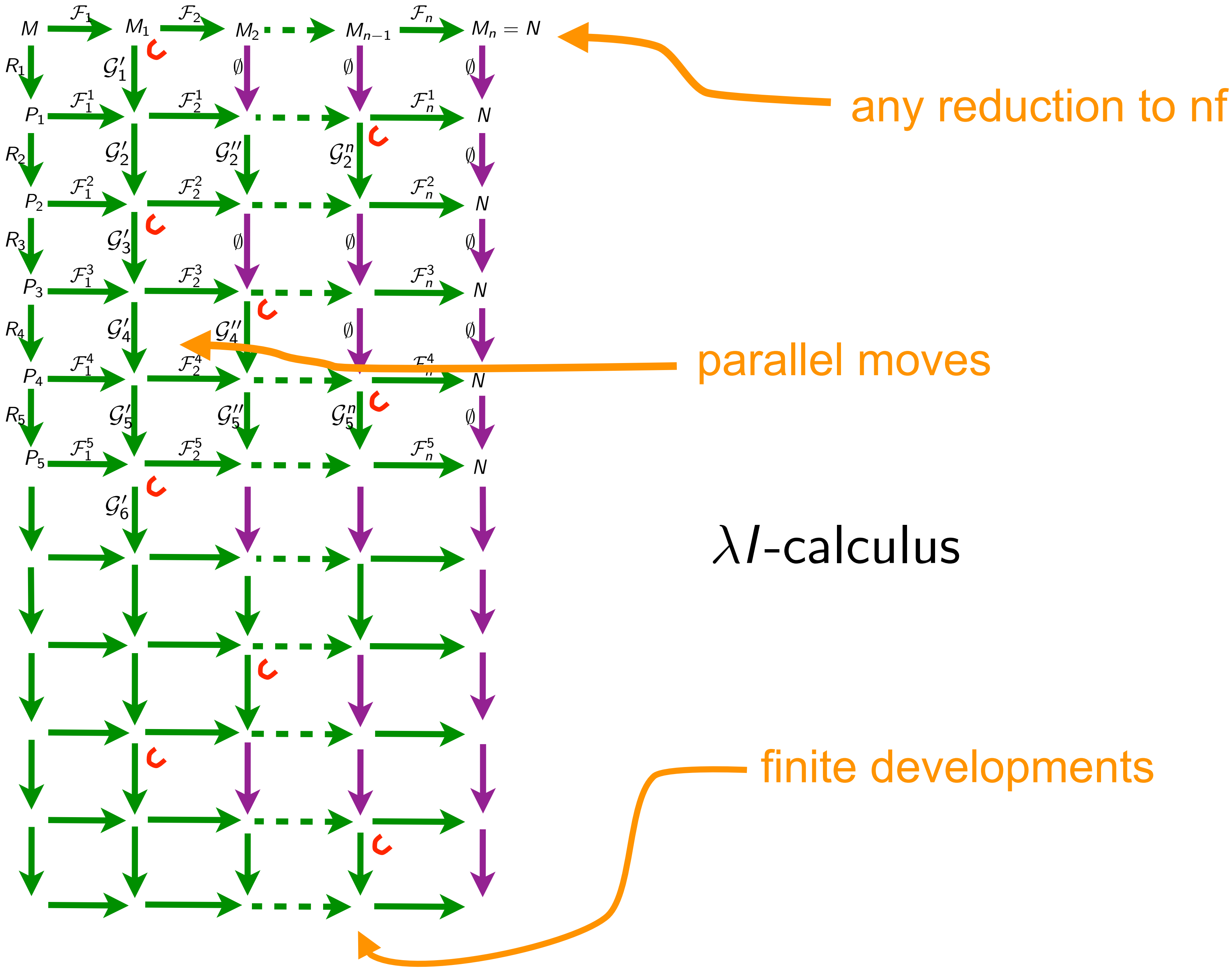
Weak vs Strong Normalisation



- true in any PTS lambda system

[conjecture Barendregt / Geuvers]

Weak vs Strong Normalisation



λI-calculus

Weak normalization in lambda-I

- innermost reduction clearly terminates (in lambda-K fst order)

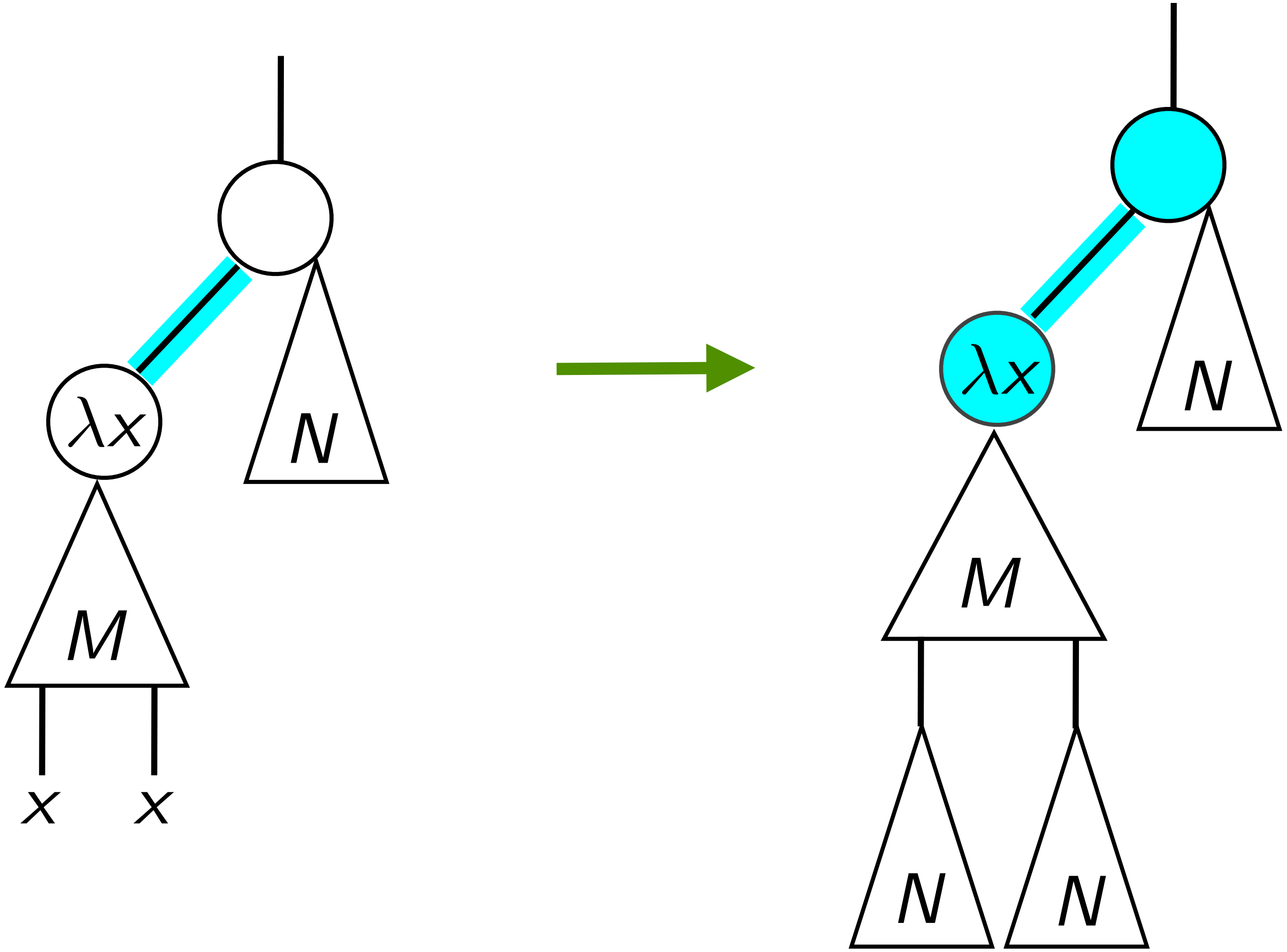
(take multiset ordering on degrees of redexes)

- weak implies strong in lambda-I

(take same argument as for standardization proof: finite developments + cube lemma)

Weak vs Strong Normalisation

- Nederpelt[72], Klop[80], Sorensen[?]



Strong Normalisation (1st order)

- why typed 1st-order calculus normalizes ?

$$(\lambda x. \cdots x N \cdots)(\lambda y.M) \rightarrow \cdots (\lambda y.M)N' \cdots$$

$\sigma \rightarrow \tau$ $\quad \sigma \quad \sigma$

creates

creation downward

creation upward

$$(\lambda x.\lambda y.M)NP \rightarrow (\lambda y.M')P$$

τ $\quad \sigma \rightarrow \tau$ $\quad \tau$

creates

$$(\lambda x.x)(\lambda y.M)N \rightarrow (\lambda y.M)N$$

$\tau \rightarrow \tau$ $\quad \tau$ $\quad \tau$

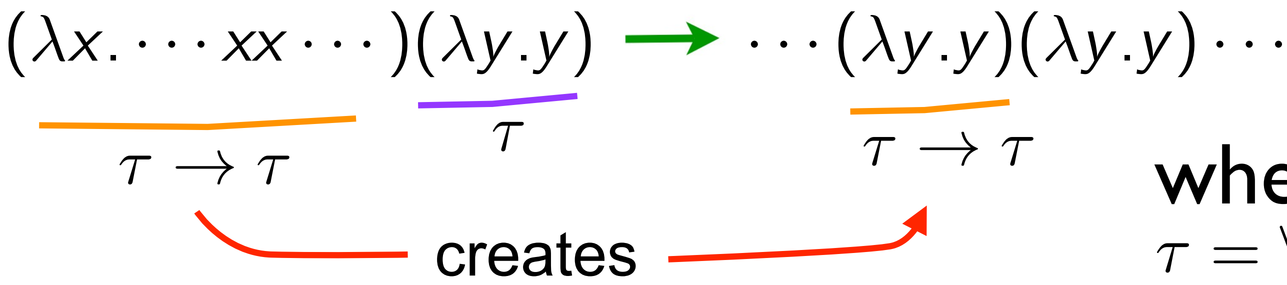
creates

- degree of a redex is type of its function part

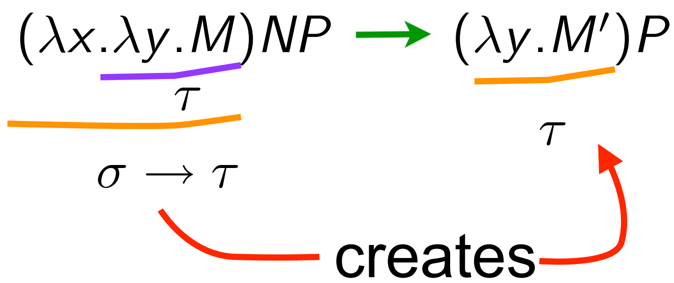
- degree strictly decreases with creation

Strong Normalisation(2nd order)

- why system F normalizes ?



where
$\tau = \forall \alpha \,.\, \alpha \rightarrow \alpha$



Strong Normalisation(2nd order)

- looking more closely at system F

$$(\lambda x. \cdots xx \cdots)(\lambda y.y) \rightarrow \cdots (\lambda y.y)(\lambda y.y) \cdots$$

$\tau \to \tau$ $\quad$ τ $\quad$ $\tau \to \tau$

creates

where
$\tau = \forall \alpha \, . \, \alpha \to \alpha$

$$(\lambda x. \cdots xx \cdots)(\lambda y.y) \rightarrow \cdots (\lambda y.y)(\lambda y.y) \cdots$$

2nd order

fst order !

also typable with

$\forall \alpha \, . \, \tau' \to \tau'$ where
$\tau' = \alpha \to \alpha$

Strong Normalisation(2nd order)

$$\Delta I \Delta I \longrightarrow I I \Delta I \longrightarrow I \Delta I \longrightarrow \Delta I \longrightarrow I I \longrightarrow I$$

$\tau \to \tau$ 1st order 1st order $\tau \to \tau$ 1st order

where

$\tau = \forall \alpha \,.\, \alpha \to \alpha$

$\Delta = \lambda x.xx$

$I = \lambda x.x$

Girard - Tait - Krivine proof

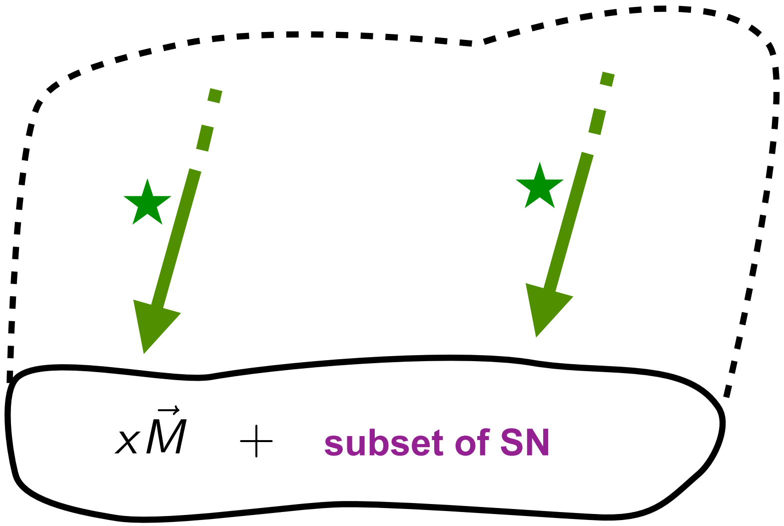
- **Definition** (**saturated sets**) $X \in \mathrm{SAT}$ iff $X \subset \mathcal{SN}$ and

(1) $\quad x\vec{M} \in X \quad$ when $\quad \vec{M} \in \vec{\mathcal{SN}}$

(2) $\quad M\{x := N\}\vec{P} \in X$ and $N \in \mathcal{SN} \quad$ implies $\quad (\lambda x.M)N\vec{P} \in X$

(1) = non emptyness **(2) = closed by SN-head-beta-expansion**

A saturated set



Girard - Tait - Krivine proof

Let $\mathcal{N}_0 = \{x\vec{M} \mid \vec{M} \in \mathcal{S}\vec{\mathcal{N}}\}$

and $X \to Y = \{M \mid N \in X \;\Rightarrow\; MN \in Y\}$

- **Fact 1** $\mathcal{N}_0 \;\subset\; \mathcal{SN} \to \mathcal{N}_0 \;\subset\; \mathcal{N}_0 \to \mathcal{SN} \;\subset\; \mathcal{SN}$
- **Fact 2** $\mathcal{SN} \in \mathrm{SAT}$

- **Lemma 1** $X, Y \in \mathrm{SAT}$ implies $X \to Y \in \mathrm{SAT}$
- **Lemma 2** $X_i \in \mathrm{SAT}$ implies $\bigcap_{i \in I} X_i \in \mathrm{SAT}$

Girard - Tait - Krivine proof

- **Semantics of types** Let $\zeta \in \mathsf{TVar} \to \mathsf{SAT}$. Then $[\![\tau]\!]_\zeta$ is

$$[\![\alpha]\!]_\zeta = \zeta(\alpha)$$

$$[\![\sigma \to \tau]\!]_\zeta = [\![\sigma]\!]_\zeta \to [\![\tau]\!]_\zeta \qquad [\![\forall\alpha.\tau]\!]_\zeta = \bigcap_{X \in \mathsf{SAT}} [\![\tau]\!]_{\zeta\{\alpha \mapsto X\}}$$

- **Corollary (1-2)** $[\![\tau]\!]_\zeta \in \mathsf{SAT}$

- **Lemma 3 (subst)** $[\![\tau\{\alpha := \sigma\}]\!]_\zeta = [\![\tau]\!]_{\zeta\{\alpha \mapsto [\![\sigma]\!]_\zeta\}}$

- **Lemma 4** Let $x_1{:}\tau_1, \ldots x_n{:}\tau_n \vdash M{:}\tau$ and $N_1 \in [\![\tau_1]\!]_\zeta, \ldots N_n \in [\![\tau_n]\!]_\zeta$
  Then $M\{x_1 := N_1, \ldots x_n := N_n\} \in [\![\tau]\!]_\zeta$

- **Corollary (4)** $\Gamma \vdash M{:}\tau$ implies $M \in \mathcal{SN}$

Girard - Tait - Krivine proof

- **Semantics of terms** Let $\rho \in \mathsf{Var} \to \Lambda$. Then

$$[\![M]\!]_\rho = M\{x_1 := \rho(x_1), \ldots x_n := \rho(x_n)\}$$

$$\rho, \zeta \models M{:}\tau \ \text{ iff } \ [\![M]\!]_\rho \in [\![\tau]\!]_\zeta$$

$$\rho, \zeta \models \Gamma \ \text{ iff } \ \rho, \zeta \models x{:}\tau \ \text{ for any } \ (x{:}\tau) \in \Gamma$$

$$\Gamma \models M{:}\tau \ \text{ iff } \ \forall \rho, \zeta \quad \rho, \zeta \models \Gamma \Rightarrow \rho, \zeta \models M{:}\tau$$

- **Lemma 3 (subst)** $[\![\tau\{\alpha := \sigma\}]\!]_\zeta = [\![\tau]\!]_{\zeta\{\alpha \mapsto [\![\sigma]\!]_\zeta\}}$
- **Lemma 4** $\Gamma \vdash M{:}\tau$ implies $\Gamma \models M{:}\tau$
- **Corollary** $\Gamma \vdash M{:}\tau$ implies $M \in \mathcal{SN}$

Simple higher-order calculus

$$M, N, A, B, \ldots ::= x \mid MN \mid \lambda x{:}A\,.\,M \mid \Pi x{:}A\,.\,B$$

$$(\lambda x{:}A\,.\,M)N \longrightarrow M\{x := N\}$$

The 2 theorems

- Confluence
- Strong normalisation in typed calculi when sorts are well-founded

Tracking redexes in untyped calculus

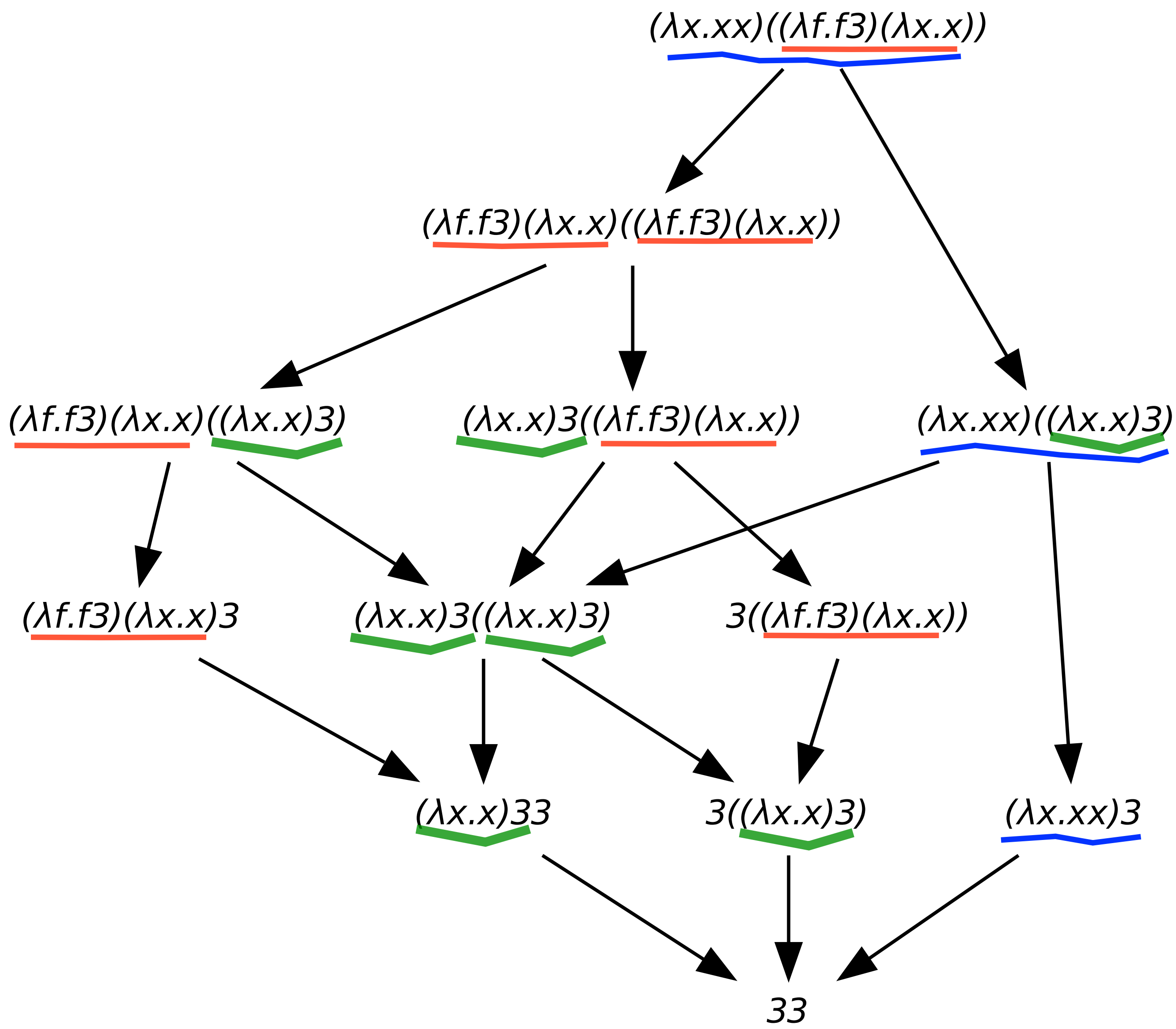
$$M, N, \ldots ::= x \mid MN \mid \lambda x \,.\, M$$

$$(\lambda x \,.\, M)N \longrightarrow M\{x := N\}$$

The 2 theorems

- Confluence
- Finite developments (cube lemma)

Redex families



- 3 redex families: **red**, **blue**, **green**.

Tracking redexes in untyped calculus

$$M, N, \ldots ::= {}^{\alpha}x \mid {}^{\alpha}(MN) \mid {}^{\alpha}(\lambda x\,.\,M)$$

$${}^{\beta}({}^{\alpha}(\lambda x\,.\,M)N) \longrightarrow {}^{\beta\lceil\alpha\rceil}M\{x := {}^{\lfloor\alpha\rfloor}N\}$$

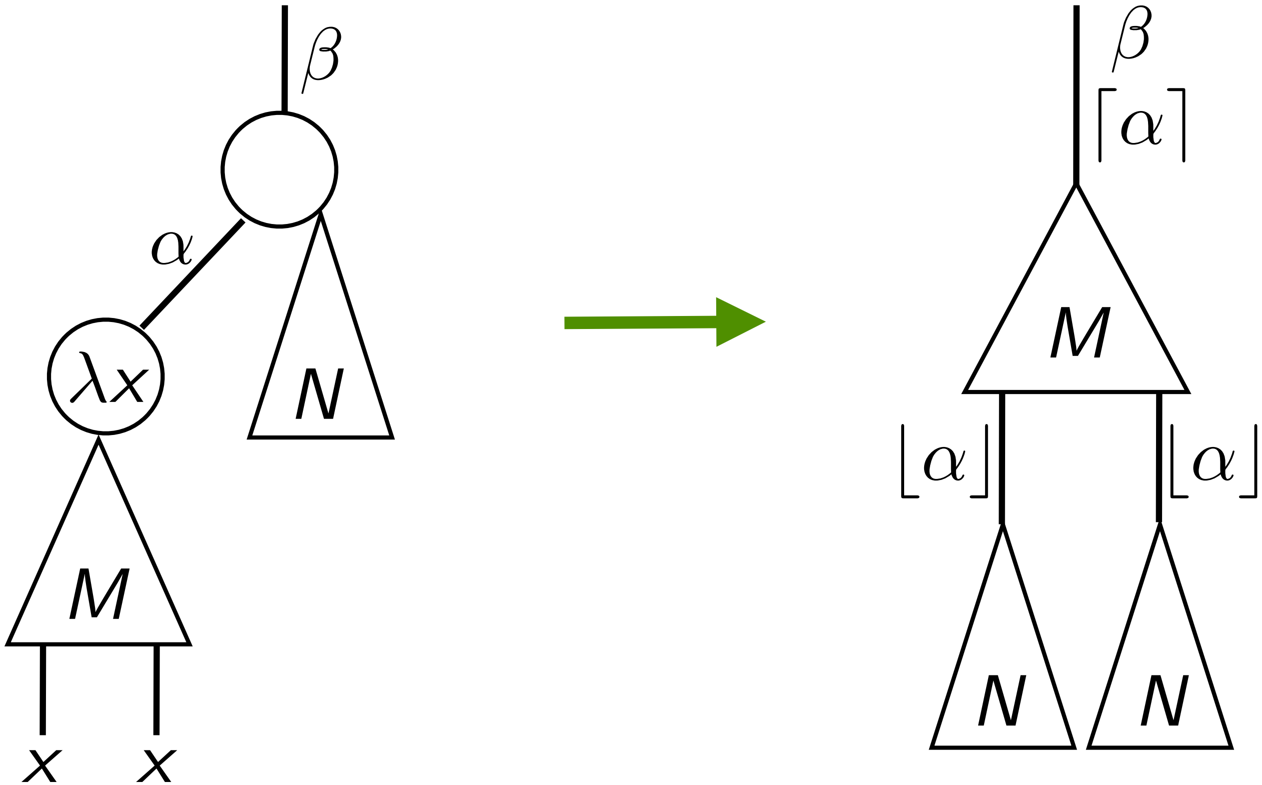
where

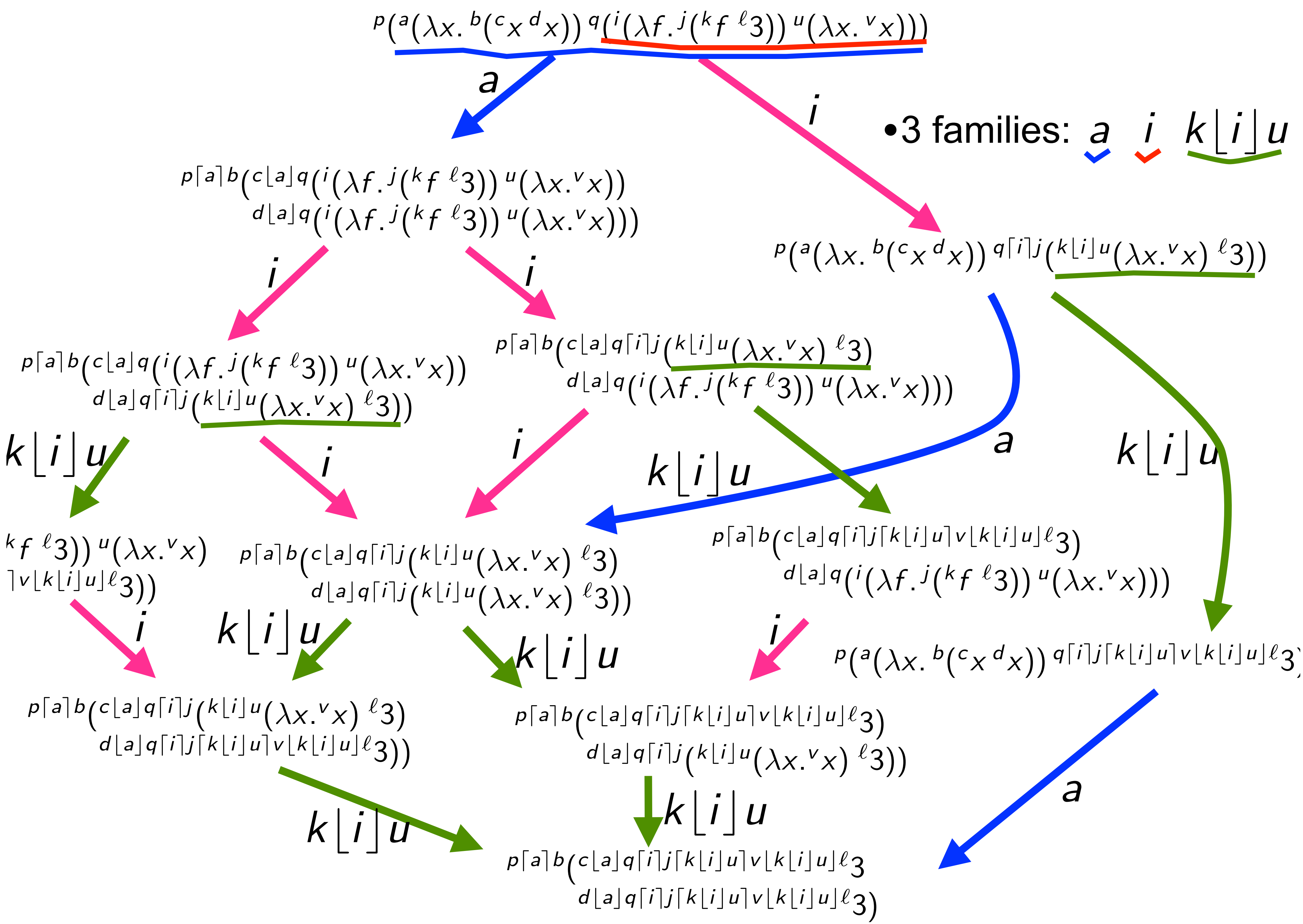
$${}^{\alpha}({}^{\beta}U) = {}^{\alpha\beta}U \quad \textbf{and} \quad {}^{\alpha}x\{x := M\} = {}^{\alpha}M$$

The 2 theorems

- Confluence (consistent names of redexes)
- Created redexes contain names of creators

Graphically





p(a(λx. b(c x d x)) q(i(λf. j(k f ℓ3)) u(λx. v x)))
a
i
•3 families: a i k⌊i⌋u
p⌈a⌉b(c⌊a⌋q(i(λf. j(k f ℓ3)) u(λx. v x)) d⌊a⌋q(i(λf. j(k f ℓ3)) u(λx. v x)))
p(a(λx. b(c x d x)) q⌈i⌉j(k⌊i⌋u(λx. v x) ℓ3))
i
i
p⌈a⌉b(c⌊a⌋q(i(λf. j(k f ℓ3)) u(λx. v x)) d⌊a⌋q⌈i⌉j(k⌊i⌋u(λx. v x) ℓ3))
p⌈a⌉b(c⌊a⌋q⌈i⌉j(k⌊i⌋u(λx. v x) ℓ3) d⌊a⌋q(i(λf. j(k f ℓ3)) u(λx. v x)))
k⌊i⌋u
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k f ℓ3)) u(λx. v x)
⌉v⌊k⌊i⌋u⌋ℓ3))
p⌈a⌉b(c⌊a⌋q⌈i⌉j(k⌊i⌋u(λx. v x) ℓ3) d⌊a⌋q⌈i⌉j(k⌊i⌋u(λx. v x) ℓ3))
p⌈a⌉b(c⌊a⌋q⌈i⌉j⌈k⌊i⌋u⌉v⌊k⌊i⌋u⌋ℓ3) d⌊a⌋q(i(λf. j(k f ℓ3)) u(λx. v x)))
i
k⌊i⌋u
k⌊i⌋u
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p(a(λx. b(c x d x)) q⌈i⌉j⌈k⌊i⌋u⌉v⌊k⌊i⌋u⌋ℓ3)
p⌈a⌉b(c⌊a⌋q⌈i⌉j(k⌊i⌋u(λx. v x) ℓ3) d⌊a⌋q⌈i⌉j⌈k⌊i⌋u⌉v⌊k⌊i⌋u⌋ℓ3))
p⌈a⌉b(c⌊a⌋q⌈i⌉j⌈k⌊i⌋u⌉v⌊k⌊i⌋u⌋ℓ3) d⌊a⌋q⌈i⌉j(k⌊i⌋u(λx. v x) ℓ3))
k⌊i⌋u
k⌊i⌋u
a
p⌈a⌉b(c⌊a⌋q⌈i⌉j⌈k⌊i⌋u⌉v⌊k⌊i⌋u⌋ℓ3 d⌊a⌋q⌈i⌉j⌈k⌊i⌋u⌉v⌊k⌊i⌋u⌋ℓ3)

Finite and infinite reductions (1/3)

- **Definition** A **reduction relative** to a set $\mathcal{F}$ of redex families is any reduction contracting redexes in families of $\mathcal{F}$.

  A **development** of $\mathcal{F}$ is any maximal relative reduction.

- **Theorem** `[Finite Developments+, 76]`
  Let $\mathcal{F}$ be a finite set of redex families.

  (1) there are no infinite reductions relative to $\mathcal{F}$,
  (2) they all finish on same term N
  (3) All developments are equivalent by permutations.

Finite and infinite reductions (2/3)

- **Corollary** An **infinite reduction** contracts an **infinite set of redex families.**

- **Corollary** The first-order typed λ-calculus strongly terminates.

Proof In first-order typed λ-calculus:

(1) residuals $R' = (\lambda x.M')N'$ of $R = (\lambda x.M)N$ keep the degree

(2) new redexes have lower degree

Tracking redexes in HO calculus

$$M, N, A, B, \ldots ::= {}^{\alpha}x \mid {}^{\alpha}(MN) \mid {}^{\alpha}(\lambda x{:}A\,.\,M) \mid {}^{\alpha}(\Pi x{:}A\,.\,B)$$

$${}^{\beta}({}^{\alpha}(\lambda x{:}A\,.\,M)N) \longrightarrow {}^{\beta\lceil\alpha,A\rceil}M\{x := {}^{\lfloor\alpha,A\rfloor}N\}$$

where

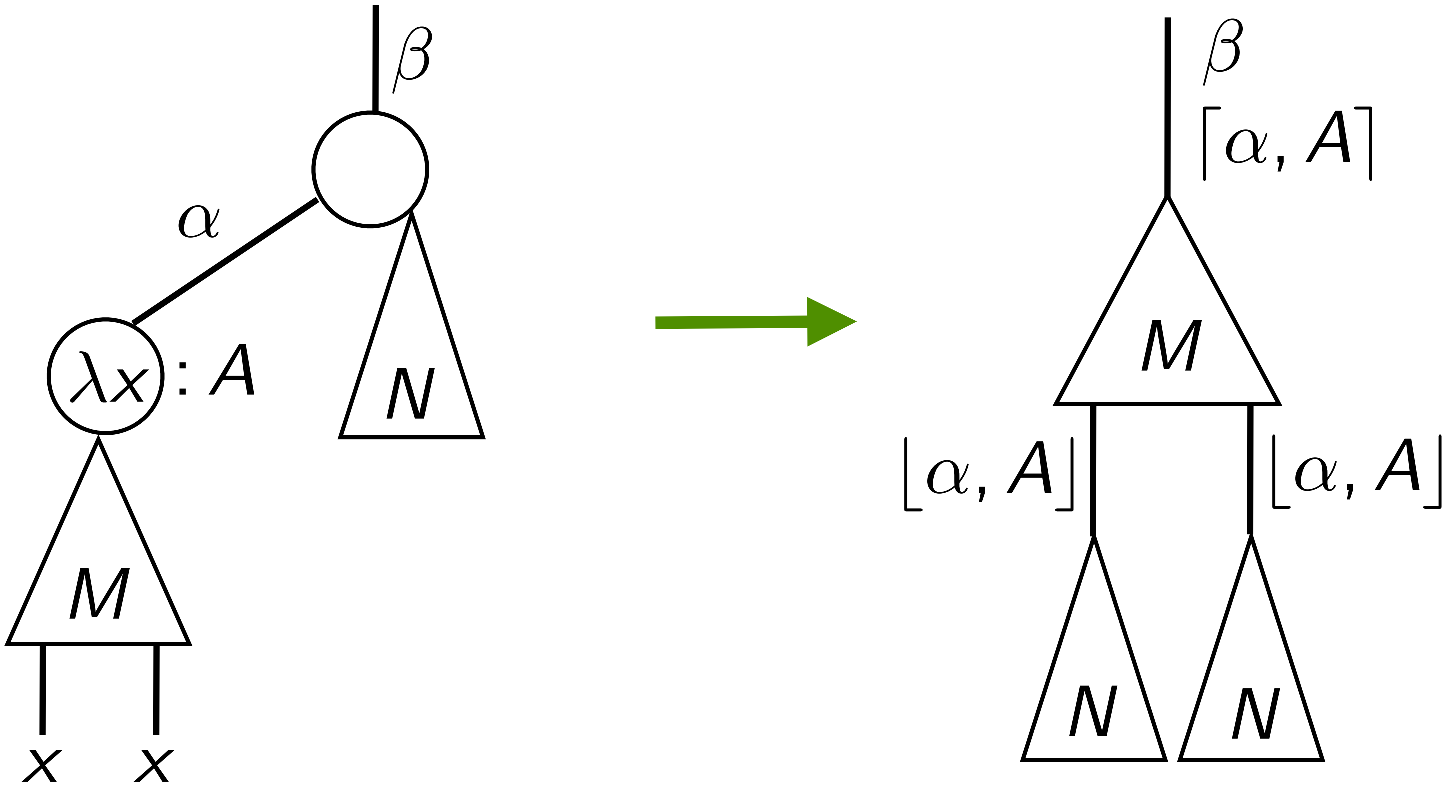
$${}^{\alpha}({}^{\beta}U) = {}^{\alpha\beta}U \quad \text{and} \quad {}^{\alpha}x\{x := M\} = {}^{\alpha}M$$

$$\text{and} \quad ({}^{\lceil\alpha,A\rceil}M)\{x := N\} = {}^{\lceil\alpha,A\{x:=N\}\rceil}M\{x := N\}$$

The 1 theorem

- Confluence

Graphically



Example

$$\Delta I \longrightarrow II \longrightarrow I \qquad \tau = \forall t\,.\,t \to t$$

$$
\begin{array}{ll}
& (\lambda x{:}\tau\,.\,x\tau x)(\Lambda t.\lambda y{:}t.y) \\
\longrightarrow & (\Lambda t.\lambda y{:}t.x)\tau(\Lambda t.\lambda y{:}t.y) \\
\longrightarrow & (\lambda y{:}\tau.x)(\Lambda t.\lambda y{:}t.y) \\
\longrightarrow & (\Lambda t.\lambda y{:}t.y)
\end{array}
$$

Example

$$\Delta I \longrightarrow II \longrightarrow I \qquad A = {}^{b}(\forall t\,.\,{}^{a}({}^{c}t \rightarrow {}^{d}t))$$

$$
\begin{array}{ll}
 & (\lambda x{:}A\,.\,x\,A'\,x)(\Lambda t.\lambda y{:}t.y) \\
\longrightarrow & (\Lambda t.\lambda y{:}t.x)A'\,(\Lambda t.\lambda y{:}t.y) \\
\longrightarrow & (\lambda y{:}A'.x)(\Lambda t.\lambda y{:}t.y) \\
\longrightarrow & (\Lambda t.\lambda y{:}t.y)
\end{array}
$$

$$
\begin{array}{ll}
 & {}^{9}({}^{4}(\lambda x{:}A\,.\,{}^{3}({}^{1}({}^{0}x\,A')\,{}^{2}x))^{8}(\Lambda t.{}^{7}(\lambda y{:}{}^{5}t.{}^{6}y))) \\
\longrightarrow & {}^{9\lceil 4,A\rceil 3}(\,{}^{1}({}^{0\lfloor 4,A\rfloor 8}(\Lambda t.{}^{7}(\lambda y{:}{}^{5}t.{}^{6}y))A')\,{}^{2\lfloor 4,A\rfloor 8}(\Lambda t.{}^{7}(\lambda y{:}{}^{5}t.{}^{6}y))) \\
\longrightarrow & {}^{9\lceil 4,A\rceil 3}(\,{}^{1\lceil 0\lfloor 4,A\rfloor 8,*\rceil 7}(\lambda y{:}\,{}^{5\lfloor 0\lfloor 4,A\rfloor 8,*\rfloor}A'.{}^{6}y)\,{}^{2\lfloor 4,A\rfloor 8}(\Lambda t.{}^{7}(\lambda y{:}{}^{5}t.{}^{6}y))) \\
\longrightarrow & 9\lceil 4,A\rceil 3\lceil 1\lceil 0\lfloor 4,A\rfloor 8,*\rceil 7,\;{}^{5\lfloor 0\lfloor 4,A\rfloor 8,*\rfloor}A'\rceil 6\lfloor 1\lceil 0\lfloor 4,A\rfloor 8,*\rceil 7,\;{}^{5\lfloor 0\lfloor 4,A\rfloor 8,*\rfloor}A'\rfloor \\
 & {}^{2\lfloor 4,A\rfloor 8}(\Lambda t.{}^{7}(\lambda y{:}{}^{5}t.{}^{6}y))
\end{array}
$$

Todo list

- Relate tracking of redexes to impredicative Girard's proof
- Find intuitive argument for SN in higher-order typed λ-calculus
- Find intuitive proof for SN in higher-order typed λ-calculus
- SN proof must always be in 3rd-order Peano logic

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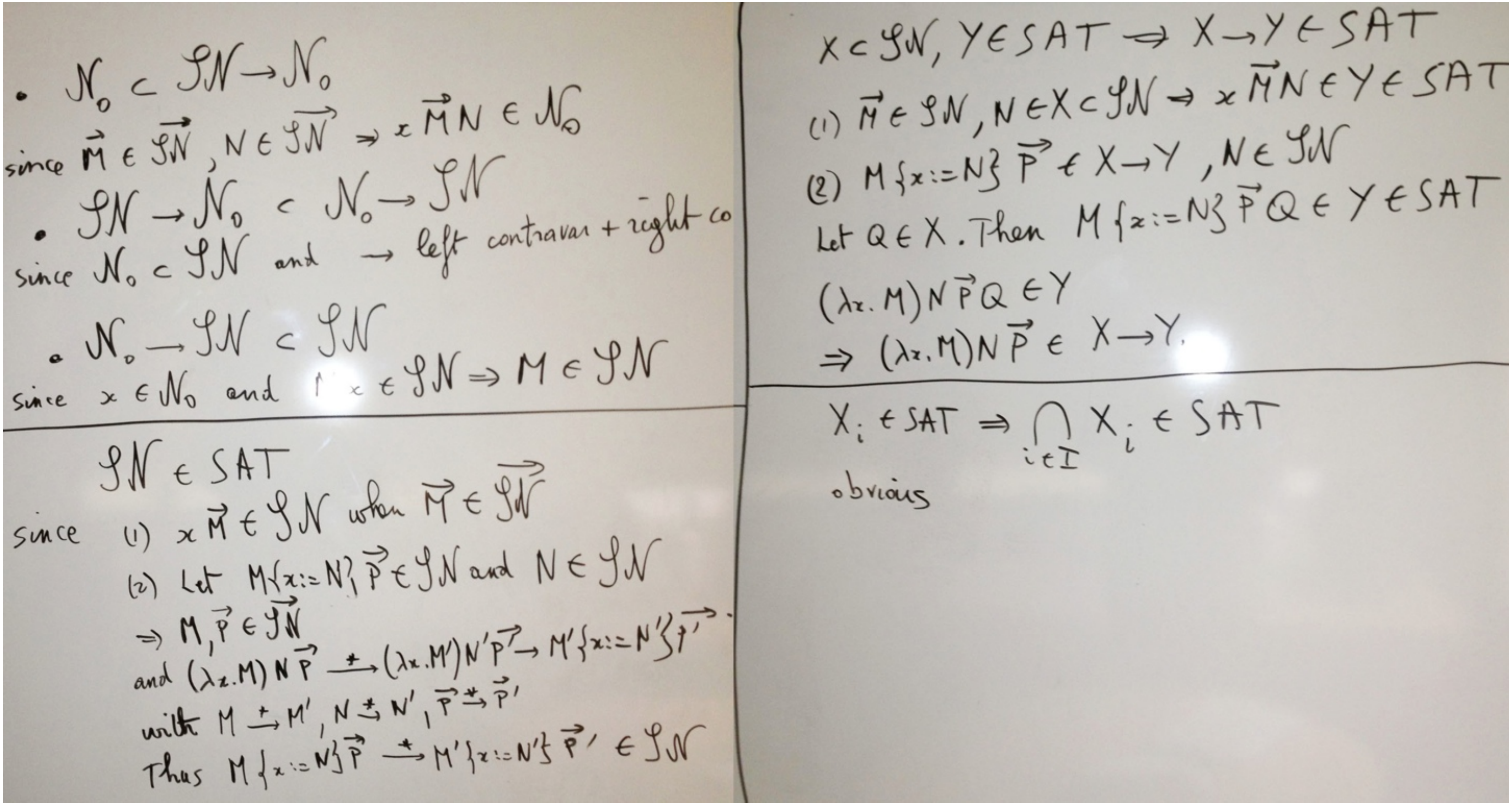
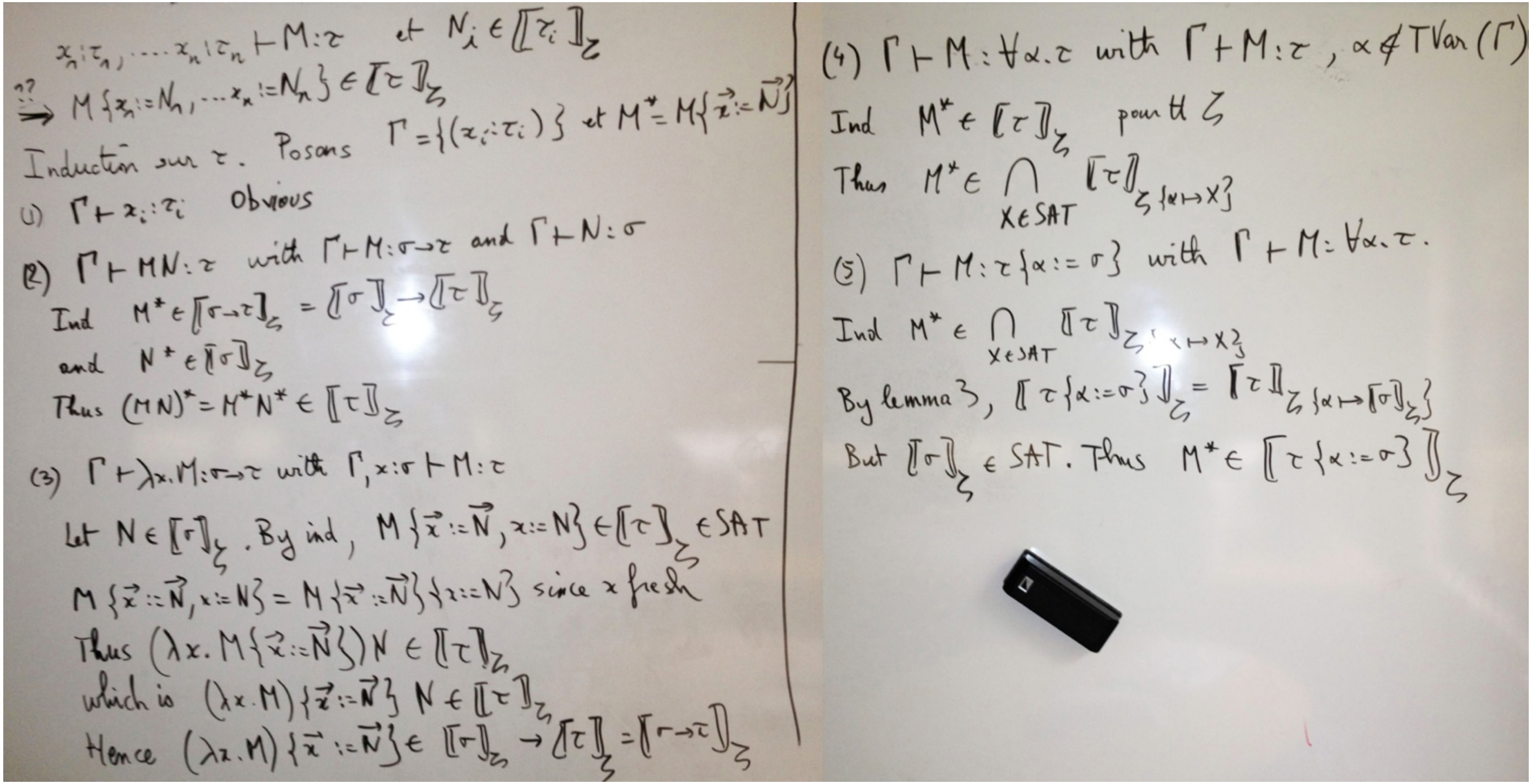
Proofs



• $\mathcal{N}_0 \subset \mathcal{SN} \to \mathcal{N}_0$
since $\vec{M} \in \vec{\mathcal{SN}}$, $N \in \mathcal{SN} \Rightarrow x\vec{M}N \in \mathcal{N}_0$
• $\mathcal{SN} \to \mathcal{N}_0 \subset \mathcal{N}_0 \to \mathcal{SN}$
since $\mathcal{N}_0 \subset \mathcal{SN}$ and $\to$ left contravar + right co
• $\mathcal{N}_0 \to \mathcal{SN} \subset \mathcal{SN}$
since $x \in \mathcal{N}_0$ and $M\,x \in \mathcal{SN} \Rightarrow M \in \mathcal{SN}$
$\mathcal{SN} \in SAT$
since (1) $x\vec{M} \in \mathcal{SN}$ when $\vec{M} \in \vec{\mathcal{SN}}$
(2) Let $M\{x:=N\}\vec{P} \in \mathcal{SN}$ and $N \in \mathcal{SN}$
$\Rightarrow M, \vec{P} \in \vec{\mathcal{SN}}$
and $(\lambda x.M)N\vec{P} \xrightarrow{*} (\lambda x.M')N'\vec{P'} \to M'\{x:=N'\}\vec{P'}$
with $M \xrightarrow{*} M'$, $N \xrightarrow{*} N'$, $\vec{P} \xrightarrow{*} \vec{P'}$
Thus $M\{x:=N\}\vec{P} \xrightarrow{*} M'\{x:=N'\}\vec{P'} \in \mathcal{SN}$
$X \subset \mathcal{SN}$, $Y \in SAT \Rightarrow X \to Y \in SAT$
(1) $\vec{M} \in \mathcal{SN}$, $N \in X \subset \mathcal{SN} \Rightarrow x\vec{M}N \in Y \in SAT$
(2) $M\{x:=N\}\vec{P} \in X \to Y$, $N \in \mathcal{SN}$
Let $Q \in X$. Then $M\{x:=N\}\vec{P}Q \in Y \in SAT$
$(\lambda x.M)N\vec{P}Q \in Y$
$\Rightarrow (\lambda x.M)N\vec{P} \in X \to Y$.
$X_i \in SAT \Rightarrow \bigcap_{i \in I} X_i \in SAT$
obvious

Proofs


x_1:τ_1, … x_n:τ_n ⊢ M:τ et N_i ∈ ⟦τ_i⟧_ζ
?⟹ M{x_1:=N_1, … x_n:=N_n} ∈ ⟦τ⟧_ζ
Induction sur τ. Posons Γ = {(x_i:τ_i)} et M* = M{x⃗:=N⃗}
(1) Γ ⊢ x_i:τ_i Obvious
(2) Γ ⊢ MN:τ with Γ ⊢ M:σ→τ and Γ ⊢ N:σ
Ind M* ∈ ⟦σ→τ⟧_ζ = ⟦σ⟧_ζ → ⟦τ⟧_ζ
and N* ∈ ⟦σ⟧_ζ
Thus (MN)* = M*N* ∈ ⟦τ⟧_ζ
(3) Γ ⊢ λx.M:σ→τ with Γ, x:σ ⊢ M:τ
Let N ∈ ⟦σ⟧_ζ. By ind, M{x⃗:=N⃗, x:=N} ∈ ⟦τ⟧_ζ ∈ SAT
M{x⃗:=N⃗, x:=N} = M{x⃗:=N⃗}{x:=N} since x fresh
Thus (λx.M{x⃗:=N⃗})N ∈ ⟦τ⟧_ζ
which is (λx.M){x⃗:=N⃗} N ∈ ⟦τ⟧_ζ
Hence (λx.M){x⃗:=N⃗} ∈ ⟦σ⟧_ζ → ⟦τ⟧_ζ = ⟦σ→τ⟧_ζ
(4) Γ ⊢ M:∀α.τ with Γ ⊢ M:τ, α ∉ TVar(Γ)
Ind M* ∈ ⟦τ⟧_ζ pour tt ζ
Thus M* ∈ ⋂_{X∈SAT} ⟦τ⟧_{ζ{α↦X}}
(5) Γ ⊢ M:τ{α:=σ} with Γ ⊢ M:∀α.τ.
Ind M* ∈ ⋂_{X∈SAT} ⟦τ⟧_{ζ, α↦X}
By lemma 3, ⟦τ{α:=σ}⟧_ζ = ⟦τ⟧_{ζ{α↦⟦σ⟧_ζ}}
But ⟦σ⟧_ζ ∈ SAT. Thus M* ∈ ⟦τ{α:=σ}⟧_ζ