The cost of usage
in the λ-calculus

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Plan

- the standardization theorem (with upper bounds)
- our result
- rigid and minimum prefixes (stability thm)
- Xi's proof (with upper bounds)
- Xi's proof revisited with live occurences

.. joint work with Andrea Asperti (LICS 2013) ..

Shortest reductions

- non effective strategies

$$
\begin{aligned}
F_m &= \lambda x.xIxx \ldots x \\
\Delta_n &= \lambda x.x\,x \ldots x \\
G_n &= \lambda y.\Delta_n(yz) \\
G'_n &= \lambda y.(yz)(yz) \ldots (yz)
\end{aligned}
$$

$$\begin{tikzcd}
 & M = F_m G_n \arrow[dl] \arrow[dr] & \\
G_n I G_n G_n \ldots G_n \arrow[d] & & F_m G'_n \arrow[d] \\
\Delta_n(Iz) G_n G_n \ldots G_n \arrow[d] & & G'_n I G'_n G'_n \ldots G'_n \arrow[d] \\
\Delta_n z G_n G_n \ldots G_n \arrow[d, dotted, "m \text{ times}"] & & (Iz)(Iz) \ldots (Iz) G'_n G'_n \ldots G'_n \arrow[d, dotted, "n \text{ times}"] \\
 {} & & {}
\end{tikzcd}$$

Standardization

Standard reductions (1/4)

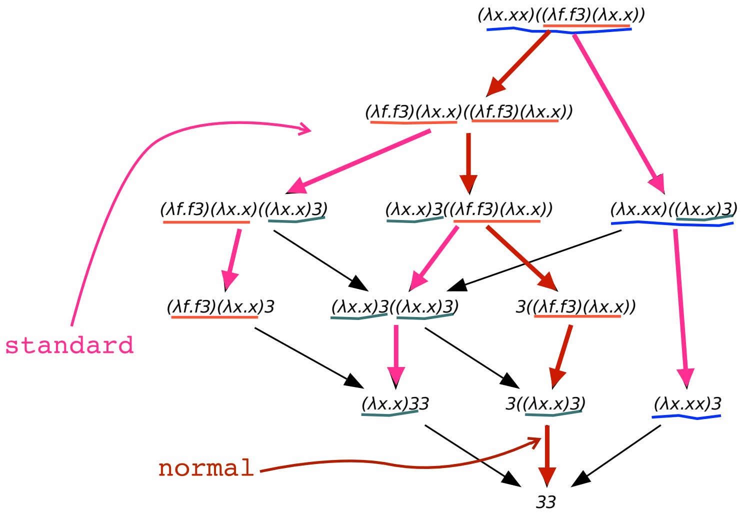
- **Definition:** The following reduction is **standard**

$$\rho : M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N$$

iff for all i and j, $i < j$, then R_j is not residual along ρ of some R'_j to the left of R_i in M_{i-1}.

- **Definition:** The leftmost-outermost reduction is also called the **normal reduction.**

Standard reductions (2/4)



(λx.xx)((λf.f3)(λx.x))

(λf.f3)(λx.x)((λf.f3)(λx.x))

(λf.f3)(λx.x)((λx.x)3)

(λx.x)3((λf.f3)(λx.x))

(λx.xx)((λx.x)3)

(λf.f3)(λx.x)3

(λx.x)3((λx.x)3)

3((λf.f3)(λx.x))

(λx.x)33

3((λx.x)3)

(λx.xx)3

33

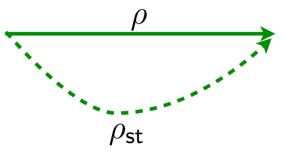
standard

normal

Standard reductions (3/4)

- **Standardization thm** `[Curry 50]`

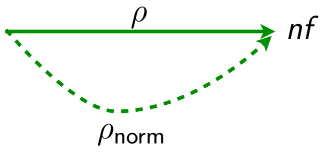
Let $M \xrightarrow{*} N$. Then $M \xrightarrow[\text{st}]{*} N$.



Any reduction can be performed outside-in and left-to-right.

- **Normalization corollary**

Let $M \xrightarrow{*} nf$. Then $M \xrightarrow[\text{norm}]{*} nf$.



Standard reductions (4/4)

- **Head reduction corollary for values**

Let $M \xrightarrow{*} V$. Then $M \xrightarrow[\text{head}]{*} \text{val}_{\text{min}}(M) \xrightarrow{*} V$

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "\text{head}"'] & V \\
\text{val}_{\text{min}}(M) \arrow[ur, dotted] &
\end{tikzcd}$$

Our result

- **Upper-bound on standard reductions** [Hongwey Xi, 99]

  Let $\ell = |\rho|$ and $\rho : M \xrightarrow{*} N$. Then $|\rho_{st}| \leq |M|^{2^\ell}$

  where $\rho_{st} : M \xrightarrow[\text{st}]{*} N$.

- **Upper-bound to normal forms** [Asperti-JJL, 13]

  Let $\ell = |\rho|$ and $\rho : M \xrightarrow{*} x$. Then $|\rho_{norm}| \leq \ell!$

  where $\rho_{norm} : M \xrightarrow[\text{norm}]{*} x$.

We gain one exponential.

Standardization proofs

- **finite developments** [Gonthier-Melliès-JJL,92]

  tricky axiomatic proof

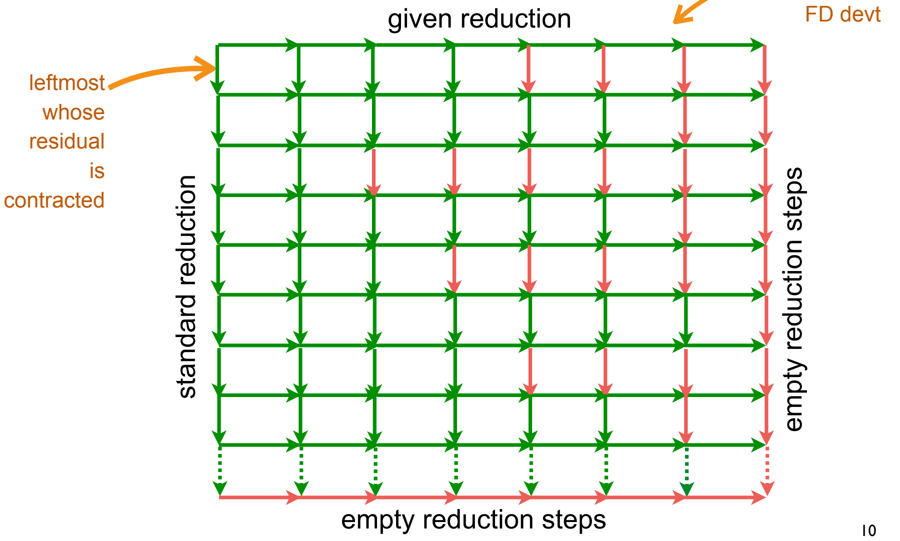
- **head normal forms** [Mitschke,80]

- **initial proof and statement** [Curry&Feys,70]

  correct statement, but proof ?

Standardization proofs

- **finite developments** [Klop, 80]



each reduction step is FD devt

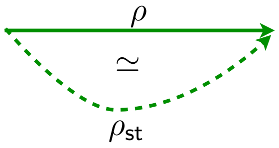
leftmost whose residual is contracted

Standard reductions (4+/4)

- **Standardization thm** [JJL 77]

  Let $\rho : M \xrightarrow{*} N$. $\exists ! \rho_{\text{st}} .\, M \xrightarrow[\text{st}]{*} N$

  and $\rho_{\text{st}} \simeq \rho$.



Standard reduction is canonical representative in permutation class.

- **λI-standardization** [Church 36]

  Standard reduction is longest in its equivalence class.

Rigid prefixes:
stability
and
multiplicity of variables

Stability (1/2)

- **Definition** `[rigid prefix]` Any rigid prefix A of M is any prefix of M where never the left of an application can reduce to an abstraction.

$$M = \Omega(\lambda x.x(Ix))(IIx)$$

$\_(\lambda x.x\_)\_$ rigid prefix of M

$$\Omega = (\lambda x.xx)(\lambda x.xx)$$

$\_(\lambda x.x\_)(\_\,Ix)$ not rigid prefix of M

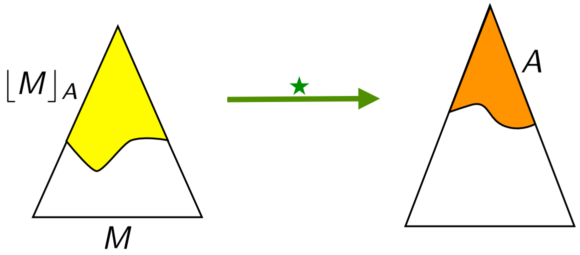
$$I = \lambda x.x$$

(rigid prefixes are finite prefixes of Berarducci trees)

- **Definition** M produces A if $M \xrightarrow{\star} N$ and A is rigid prefix of N.

Stability (2/2)

- **Theorem** `[stability]` For any rigid prefix A produced by M, there is a unique minimal prefix $\lfloor M \rfloor_A$ of M producing A.



- **Fact** `[monotony]` Let M produce A rigid and $M \xrightarrow{\star} N$. Then N produces A.

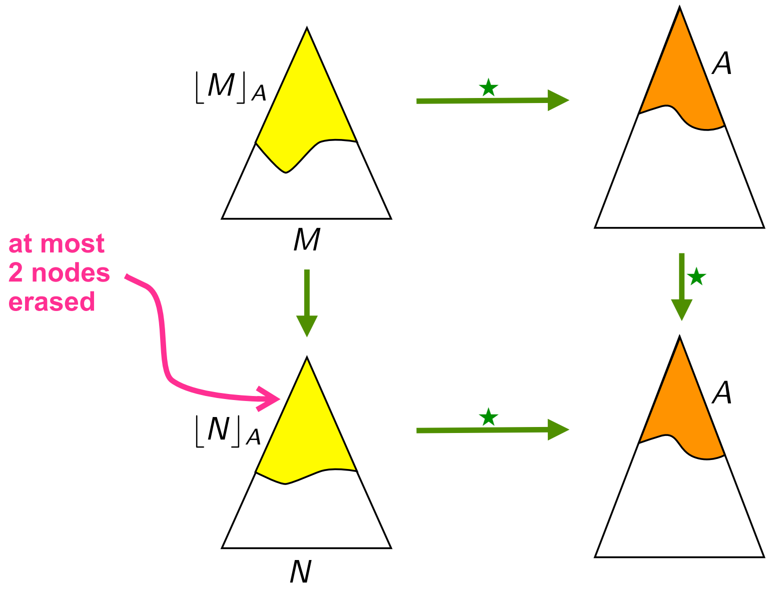
Slow consumption (1/2)

- **Lemma 1** `[slow consumption]` Let M produce A rigid and $M \rightarrow N$. Then $|\lfloor N \rfloor_A| \geq |\lfloor M \rfloor_A| - 2$.

i.e. $|\lfloor M \rfloor_A|_@ \leq 1 + |\lfloor N \rfloor_A|_@$ where $|P|_@$ is the applicative size of P (its number of application nodes).

- **Corollary** Let $\rho : M \xrightarrow{\star} N$ and A be rigid prefix of N. Then $|\lfloor M \rfloor_A|_@ \leq |\rho| + |A|_@$.

Slow consumption (2/2)



Standardization

Multiplicity of variables

- **Definition** Let M produce A rigid. An occurrence of x is live for A if it belongs to $\lfloor M \rfloor_A$.

Let $m_A(x)$ be the number of live occurrences of x in M.

We pose $m_A(R) = m_A(x)$ when $R = (\lambda x.M)N$.

- **Lemma 2** [upper bound on live multiplicity]

Let $\rho : M \xrightarrow{\star} N$ and A rigid prefix of N. Then

$m_A(x) \leq |\rho| + |A|_{@} + 1$ for any variable x in M.

Xi's proof of standardization (1/2)

- **Lemma** [reordering of head redexes] H is residual of H'. Then

$$\begin{tikzcd}
M \arrow[r, "\rho", "\mathsf{st}"'] \arrow[d, dashed, "H'"'] & N = \lambda\vec{x}.(\lambda x.V)W\vec{N} \arrow[d, "H"'] \\
M' \arrow[r, dashed, "\rho'", "\mathsf{st}"'] & N' = \lambda\vec{x}.V\{x := W\}\vec{N}
\end{tikzcd}$$

with $|\rho'| \leq \lceil 1, m(H) \rceil . |\rho|$

Proof Easy since $M = \lambda\vec{x}.(\lambda x.T)U\vec{M}$ and $\rho = \rho_T \rho_U \rho_1 \cdots \rho_n$.
And ρ' is disjoint intermix of ρ_T, several ρ_U, followed by ρ_i's.
Thus $|\rho'| = |\rho_T| + m(H).|\rho_U| + \sum_i |\rho_i|$

Xi's proof of standardization (2/2)

- **Corollary**

$$\begin{tikzcd} M \arrow[r, "\rho", "\text{st}"'] \arrow[dr, dashed, "\rho'", "\text{st}"'] & N \arrow[d, "R"] \\ & N' \end{tikzcd}$$

with $|\rho'| \leq 1 + \lceil 1, m(R) \rceil . |\rho|$

Proof
By induction on pair $(|\rho|, |M|)$. Cases on ρR contracting head redex or not + previous lemma.

Xi's proof of standardization (2/2)

- **Theorem** [standardization with upper bounds]

Let $M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N$

Then there is ρ standard from M to N such that

$|\rho| \leq (1 + \lceil 1, m(R_2) \rceil)(1 + \lceil 1, m(R_3) \rceil) \cdots (1 + \lceil 1, m(R_n) \rceil)$

Proof By induction on the length n of reduction from M to N.

Proof of our upper bound (1/2)

- **Theorem** [standardization with upper bounds]

Let $M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N$
and A be rigid prefix of N.

Then there is ρ standard from M to N' such that

$|\rho| \leq (1 + \lceil 1, m_A(R_2) \rceil)(1 + \lceil 1, m_A(R_3) \rceil) \cdots (1 + \lceil 1, m_A(R_n) \rceil)$

and A is rigid prefix of N'.

Proof of our upper bound (2/2)

- **Corollary 1** Let $\rho : M \xrightarrow{*} N$ and A be rigid prefix of N.

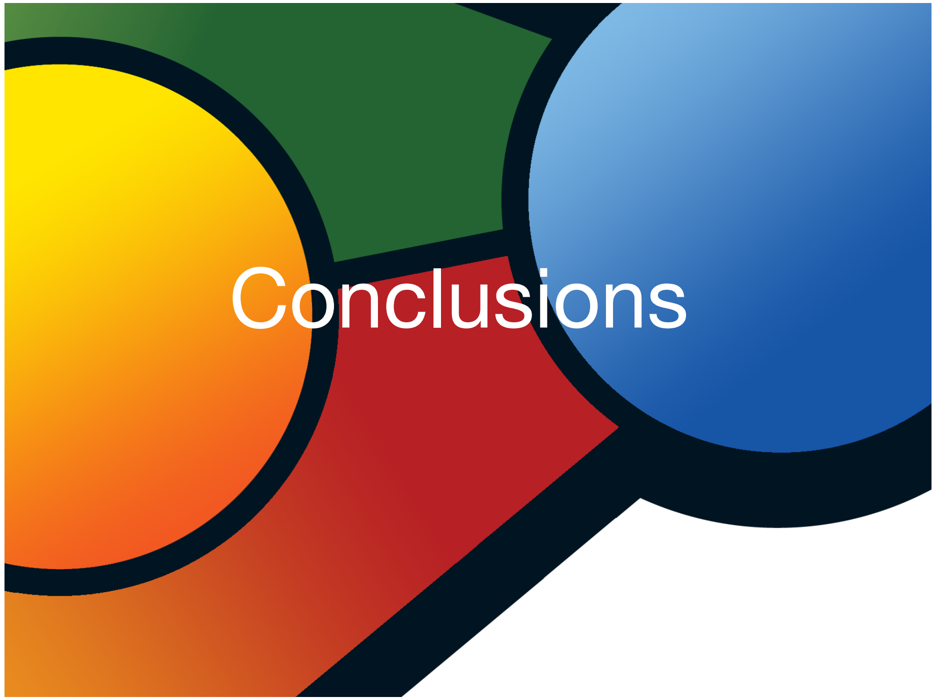
Then there is ρ_{st} standard such that:

$$|\rho_{st}| \leq \frac{(|\rho| + |A|_@)!}{(1 + |A|_@)!}$$

Proof Simple calculation with lemma 2 and previous thm.

- **Corollary 2** Let $\rho_{st} : M \xrightarrow{*} x$ be standard reduction.

Then $|\rho_{st}| \leq |\rho|!$ where ρ is shortest reduction from M to x.



Conclusions

Conclusion

- terms are easy to grow in the λ-calculus
- but take time to consume terms

→ there is a need for sharing

- back to earth …. and higher-order functional languages