MPRI Concurrency (Cours 2-3)
Midterm Exam Solutions, 2005-2006
25 Jan 2006
James.Leifer@inria.fr

Quick review of CCS operational equivalences

Strong simulation: a relation $\mathcal{R}$ is a strong simulation if for all $(P, Q) \in \mathcal{R}$ and $P \xrightarrow{\alpha} P'$, there exists Q' such that $Q \xrightarrow{\alpha} Q'$ and $(P', Q') \in \mathcal{R}$.

Strong bisimilarity: $\sim$ is the largest strong bisimulation.

Weak simulation: a relation $\mathcal{R}$ is a weak simulation if for all $(P, Q) \in \mathcal{R}$ we have:

1. if $P \xrightarrow{\tau} P'$ then there exists Q' such that $Q \xrightarrow{\tau}{}^* Q'$ and $(P', Q') \in \mathcal{R}$.
2. if $P \xrightarrow{\alpha} P'$ and $\alpha \neq \tau$ then there exists Q' such that $Q \xrightarrow{\tau}{}^* \xrightarrow{\alpha} \xrightarrow{\tau}{}^* Q'$ and $(P', Q') \in \mathcal{R}$.

Weak bisimilarity (also known as bisimilarity, also known as observational equivalence): $\approx$ is the largest weak bisimulation.

Observational congruence: $\approxeq$ is the largest symmetric relation satisfying the following property: if $P \approxeq Q$ and $P \xrightarrow{\alpha} P'$ then there exists Q' such that $Q \xrightarrow{\tau}{}^* \xrightarrow{\alpha} \xrightarrow{\tau}{}^* Q'$ and $P' \approx Q'$.

Note that $(\sim) \subseteq (\approxeq) \subseteq (\approx)$. We will freely use these inclusions throughout.

Question 1

1. We show that $(\sim)$, the strongest of the three relations, holds.

   Consider $\mathcal{R} = \{(C,\ A \mid B)\}$. The only transitions of C are to itself with labels a and b: $C \xrightarrow{a} C$ and $C \xrightarrow{b} C$. Likewise, the only transitions of $A|B$ are to itself with labels a and b: $A|B \xrightarrow{a} A|B$ and $A \mid B \xrightarrow{b} A \mid B$. Therefore $\mathcal{R}$ is bisimulation.

2. $(\sim)$ does not hold. Take $P = a$. Then the second process has the transition $\xrightarrow{a}$ but the first doesn't.

   $(\approxeq)$ does hold (hence so does $(\approx)$). We consider the labelled transitions of each side. We will rely once on the rule

   $$Q \approx \tau.Q \tag{1}$$

   and the fact that $\approx$ is closed by parallel composition.

   - The only transition of first process is $\tau.(P \mid P) \xrightarrow{\tau} P \mid P$. The second process can match this transition to the identical target $P \mid \tau.P \xrightarrow{\tau} P \mid P$.
   - There are two classes of transitions for the second process.

     Case $P \mid \tau.P \xrightarrow{\tau} P \mid P$: As seen just above the first process can match this transition to the identical target.

     Case $P \mid \tau.P \xrightarrow{\alpha} P' \mid \tau.P$ where $P \xrightarrow{\alpha} P'$: The first process can match this transition with $\tau.(P \mid P) \xrightarrow{\tau} \xrightarrow{\alpha} P' \mid P$. By (1), $P' \mid \tau.P \approx P' \mid P$, as desired.

3. None of the relations holds. Suppose for contradiction that $(\approx)$ held. Take $P = Q = x$. The first process may undergo two τ transitions and reach a deadlocked state:

$$\boldsymbol{\nu}a.(\boldsymbol{\nu}b.(\overline{a} \mid \overline{b} \mid a.b.P \mid b.a.Q)) \xrightarrow{\tau}\xrightarrow{\tau} \boldsymbol{\nu}a.(\boldsymbol{\nu}b.(\mathbf{0} \mid \mathbf{0} \mid b.P \mid a.Q)) \sim \mathbf{0}$$

By hypothesis the second process can match this state with a sequence of 0 or more τ transitions; only two target states are possible:

$$\tau.x + \tau.x \qquad\qquad x$$

but neither is deadlocked since both can do $\xrightarrow{\tau}{}^{*}\xrightarrow{x}$, a contradiction.

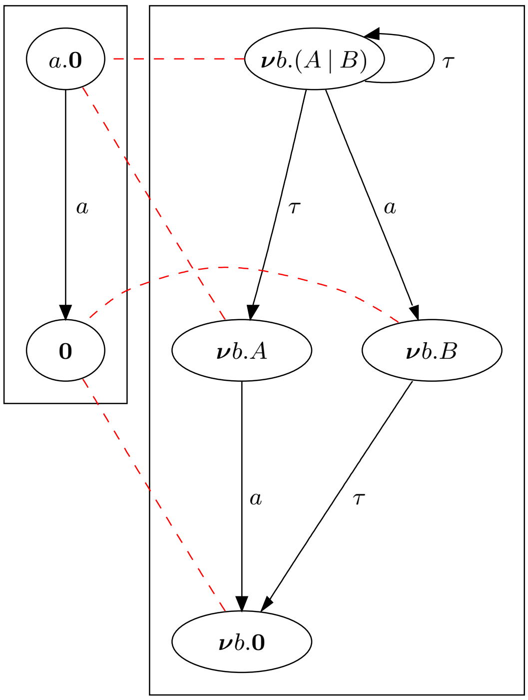
Note: the question should have had the side condition $\{a, b\} \cap (\mathsf{fn}(P) \cup \mathsf{fn}(Q)) = \varnothing$. In the absence of this side condition, there is a simpler counterexample. Take $P = a$. Then the second process can do $\xrightarrow{\tau}\xrightarrow{a}$ but the first process can never have an a transition.

4. Trick question! None of the relations holds. Take $a = b$. Then $a.\mathbf{0}$ has an a transition but the other process is weakly bisimilar to $\mathbf{0}$.

Let us assume that the examiner meant to include the side condition $a \neq b$. Then we can answer the question as follows...

The relation $\approxeq$ doesn't hold (hence neither does $\sim$). This is because $\boldsymbol{\nu}b.(A \mid B)$ can do a τ transition to itself, but $a.\mathbf{0}$ has no τ transitions.

The relation $\approx$ does hold, as illustrated by the red dotted lines relating the states of two reduction graphs. Note that we show all the states up to structural equivalence.

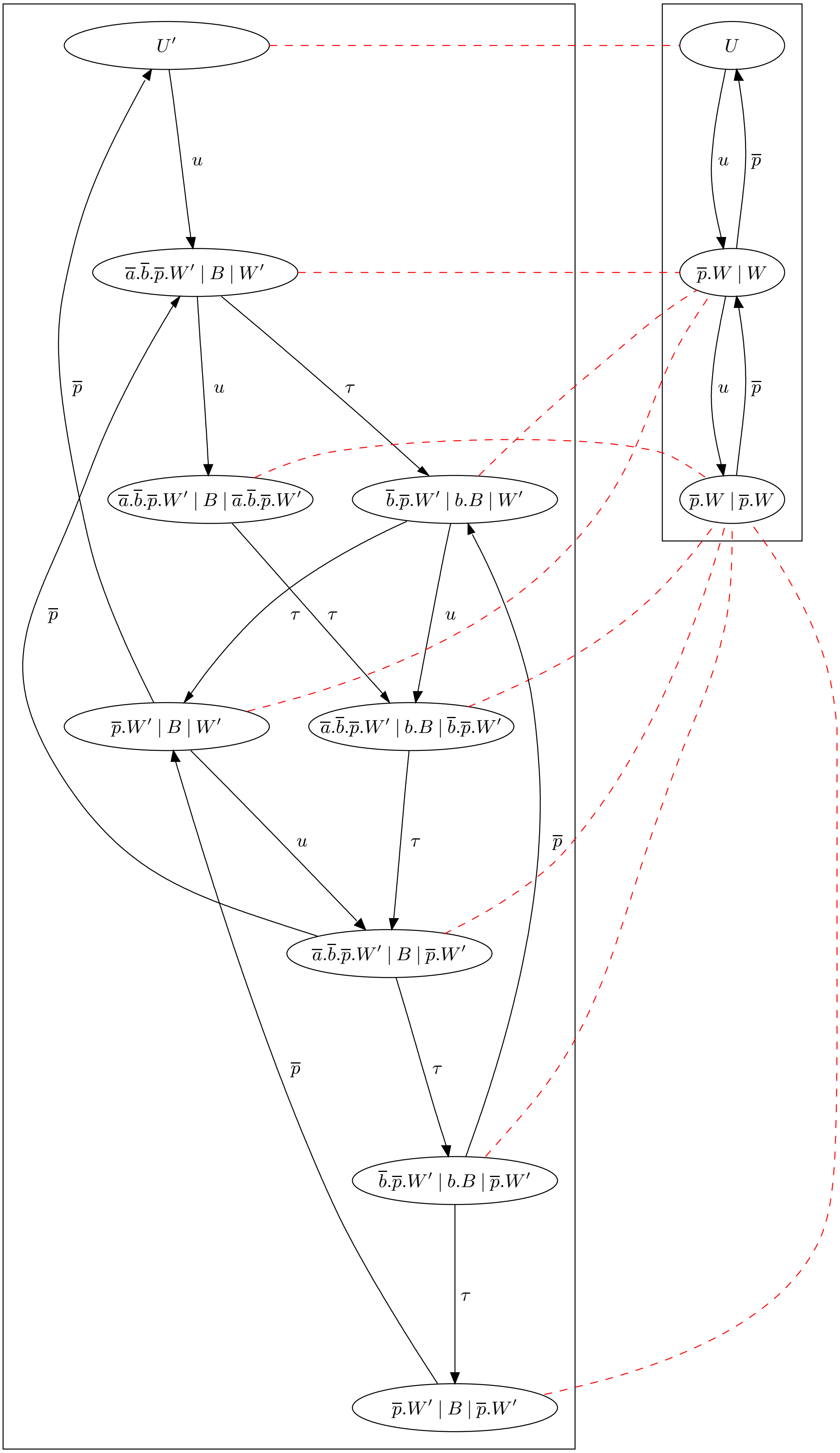


5. None of the relations holds. Suppose for contradiction that $(\approx)$ held. We have $A \xrightarrow{\tau} B$. By hypothesis, $a.\mathbf{0} \approx B$, but this is impossible because $a.\mathbf{0}$ can have an a transition while $B \approx \mathbf{0}$, a contradiction.

Question 2

They are weakly bisimilar, as illustrated by the red dotted lines relating the states of two reduction graphs. Since neither U nor U' have any initial τ transitions, $U \approxeq U'$, as desired.

Note that we omit writing the new bindings in all states in the complex graph in order to save space. Also, we show all the states up to structural equivalence.


U'
u
$\overline{a}.\overline{b}.\overline{p}.W' \mid B \mid W'$
u
τ
$\overline{p}$
$\overline{a}.\overline{b}.\overline{p}.W' \mid B \mid \overline{a}.\overline{b}.\overline{p}.W'$
$\overline{b}.\overline{p}.W' \mid b.B \mid W'$
$\overline{p}$
τ
τ
u
$\overline{p}.W' \mid B \mid W'$
$\overline{a}.\overline{b}.\overline{p}.W' \mid b.B \mid \overline{b}.\overline{p}.W'$
u
τ
$\overline{p}$
$\overline{a}.\overline{b}.\overline{p}.W' \mid B \mid \overline{p}.W'$
$\overline{p}$
τ
$\overline{b}.\overline{p}.W' \mid b.B \mid \overline{p}.W'$
τ
$\overline{p}.W' \mid B \mid \overline{p}.W'$
U
u
$\overline{p}$
$\overline{p}.W \mid W$
u
$\overline{p}$
$\overline{p}.W \mid \overline{p}.W$

Question 3

We shall use the following name vectors ("left", "right", "mid") throughout:

$$\begin{aligned}\vec{l} &= \langle empty, in, out\rangle \\ \vec{r} &= \langle e, i, o\rangle \\ \vec{m} &= \langle e', i', o'\rangle\end{aligned}$$

Then the concatenation of cells can be easily written as:

$$P \triangleright Q = (\vec{l}, \vec{r}).\boldsymbol{\nu}\vec{m}.(P\langle \vec{l}, \vec{m}\rangle \mid Q\langle \vec{m}, \vec{r}\rangle)$$

1. For $C' \triangleright C$, we calculate as follows:

$$\begin{array}{lll} & (C' \triangleright C)\langle \vec{l}, \vec{r}\rangle & \\ = & \boldsymbol{\nu}\vec{m}.(C'\langle \vec{l}, \vec{m}\rangle \mid C\langle \vec{m}, \vec{r}\rangle) & \text{definition of } \triangleright \\ = & \boldsymbol{\nu}\vec{m}.((o'.C\langle \vec{l}, \vec{m}\rangle + \overline{e'}.E\langle \vec{l}, \vec{m}\rangle) \mid (\overline{o'}.C'\langle \vec{m}, \vec{r}\rangle + i'.(C \triangleright C)\langle \vec{m}, \vec{r}\rangle)) & \text{definition of } C' \text{ and } C \\ \xrightarrow{\tau} & \boldsymbol{\nu}\vec{m}.(C\langle \vec{l}, \vec{m}\rangle \mid C'\langle \vec{m}, \vec{r}\rangle) & \text{synchronisation on } o' \\ = & (C \triangleright C')\langle \vec{l}, \vec{r}\rangle & \text{definition of } \triangleright \end{array}$$

Since the transition shown above is the only one possible (all others are prevented by the outermost new binding), we conclude $C' \triangleright C \approx C \triangleright C'$, as desired.

For $C' \triangleright E$, we calculate as follows:

$$\begin{array}{lll} & (C' \triangleright E)\langle \vec{l}, \vec{r}\rangle & \\ = & \boldsymbol{\nu}\vec{m}.(C'\langle \vec{l}, \vec{m}\rangle \mid E\langle \vec{m}, \vec{r}\rangle) & \text{definition of } \triangleright \\ = & \boldsymbol{\nu}\vec{m}.((o'.C\langle \vec{l}, \vec{m}\rangle + \overline{e'}.E\langle \vec{l}, \vec{m}\rangle) \mid (e'.E\langle \vec{m}, \vec{r}\rangle + i'.(C \triangleright E)\langle \vec{m}, \vec{r}\rangle)) & \text{definition of } C' \text{ and } E \\ \xrightarrow{\tau} & \boldsymbol{\nu}\vec{m}.(E\langle \vec{l}, \vec{m}\rangle \mid E\langle \vec{m}, \vec{r}\rangle) & \text{synchronisation on } e' \\ = & (E \triangleright E)\langle \vec{l}, \vec{r}\rangle & \text{definition of } \triangleright \end{array}$$

Since the transition shown above is the only one possible (all others are prevented by the outermost new binding), we conclude $C' \triangleright E \approx E \triangleright E$, as desired.

For $E \triangleright E$, we calculate as follows:

$$\begin{array}{lll} & (E \triangleright E)\langle \vec{l}, \vec{r}\rangle & \\ = & \boldsymbol{\nu}\vec{m}.(E\langle \vec{l}, \vec{m}\rangle \mid E\langle \vec{m}, \vec{r}\rangle) & \text{definition of } \triangleright \\ = & \boldsymbol{\nu}\vec{m}.((empty.E\langle \vec{l}, \vec{m}\rangle + in.(C \triangleright E)\langle \vec{l}, \vec{m}\rangle) \mid E\langle \vec{m}, \vec{r}\rangle) & \text{definition of } C' \\ \sim & empty.\boldsymbol{\nu}\vec{m}.(E\langle \vec{l}, \vec{m}\rangle \mid E\langle \vec{m}, \vec{r}\rangle) + in.\boldsymbol{\nu}\vec{m}.((C \triangleright E)\langle \vec{l}, \vec{m}\rangle \mid E\langle \vec{m}, \vec{r}\rangle) & \text{expansion} \\ \sim & empty.(E \triangleright E)\langle \vec{l}, \vec{r}\rangle + in.(C \triangleright (E \triangleright E))\langle \vec{l}, \vec{r}\rangle & \text{definition of } \triangleright \text{ and associativity} \end{array}$$

Therefore E and $E \triangleright E$ satisfy the same guarded recurrences, hence $E \sim E \triangleright E$, as desired.

2. We now show that $C' \triangleright C^{(k)} \approx C^{(k)}$ for $0 \leq k$. We induct on k.

Base case: We calculate:

$$\begin{array}{lll} & C' \triangleright C^{(0)} & \\ = & C' \triangleright E & \text{definition} \\ \approx & E \triangleright E & \text{previous exercise} \\ \sim & E & \text{previous exercise} \\ = & C^{(0)} & \text{definition} \end{array}$$

Since $(\sim) \subseteq (\approx)$, we have $C' \triangleright C^{(0)} \approx C^{(0)}$ as desired.

Step case: We calculate:

$$\begin{array}{lll}
& C' \triangleright C^{(k+1)} & \\
= & C' \triangleright (C \triangleright C^{(k)}) & \text{definition} \\
\sim & (C' \triangleright C) \triangleright C^{(k)} & \text{associativity} \\
\approx & (C \triangleright C') \triangleright C^{(k)} & \text{previous ex; also } \approx \text{ closed by } \triangleright \\
\approx & C \triangleright C^{(k)} & \text{inductive hypothesis} \\
= & C^{(k+1)} & \text{definition}
\end{array}$$

Since $(\sim) \subseteq (\approx)$, we have $C' \triangleright C^{(k)} \approx C^{(k)}$ as desired.

3. First we consider the base case:

$$\begin{array}{lll}
& C^{(0)}\langle \vec{l}, \vec{r} \rangle & \\
= & E\langle \vec{l}, \vec{r} \rangle & \text{definition} \\
= & \mathit{empty}.E\langle \vec{l}, \vec{r} \rangle + \mathit{in}.(C \triangleright E)\langle \vec{l}, \vec{r} \rangle & \text{definition} \\
= & \mathit{empty}.C^{(0)}\langle \vec{l}, \vec{r} \rangle + \mathit{in}.C^{(1)}\langle \vec{l}, \vec{r} \rangle & \text{definition}
\end{array}$$

Then we consider the other case:

$$\begin{array}{lll}
& C^{(k+1)}\langle \vec{l}, \vec{r} \rangle & \\
= & (C \triangleright C^{(k)})\langle \vec{l}, \vec{r} \rangle & \text{definition} \\
\sim & \overline{\mathit{out}}.(C' \triangleright C^{(k)})\langle \vec{l}, \vec{r} \rangle + \mathit{in}.((C \triangleright C) \triangleright C^{(k)})\langle \vec{l}, \vec{r} \rangle & \text{expansion} \\
\sim & \overline{\mathit{out}}.C^{(k)}\langle \vec{l}, \vec{r} \rangle + \mathit{in}.((C \triangleright C) \triangleright C^{(k)})\langle \vec{l}, \vec{r} \rangle & \text{previous exercise} \\
\sim & \overline{\mathit{out}}.C^{(k)}\langle \vec{l}, \vec{r} \rangle + \mathit{in}.C^{(k+2)}\langle \vec{l}, \vec{r} \rangle & \text{associativity and definition of } C^{(k+2)}
\end{array}$$

We have show that the families $C^{(k)}$ and B_k satisfy the same guarded recurrences, hence $C^{(k)} \approx B_k$ for all $0 \leq k$.